[11:00-11:05] Anti-aliasing in computer graphics

When rendering a 3d scene, it's possible for the spatial resolution of the geometry or textures to exceed the sampling rate (resolution or number of pixels). A low-pass anti-aliasing filter can be applied to reduce these artifacts. Example.

[11:05-11:30] Differences between discrete-time and continuous time:

$$x[n] = x(t)|_{t=nT_s}$$

Continuous-time	Discrete-time
Time axis	
Real-valued time axis	Integer valued time axis
Periodicity	
Periodicity must align with integer grid. See handout D. For two-sided cosine: $x[n] = \cos(\omega_0 n)$ where $\omega_0 = 2\pi\frac{f_0}{fs} = 2\pi\frac{N}{L}$ where N and L are relatively prime integers ,f_0 is the frequency, and f_s is the sampling rate. For periodicity, we require $$x[n + N_0] = \cos\left(2\pi\frac{N}{L}n\right) = x[n]$$ So the discrete-time cosine is periodic when N and L are integers. The smallest value of N_0 is L. If f_0/f_s is irrational (meaning it cannot be represented as a ratio of two integers), then the discrete-time cosine is not periodic.	
Frequency Domain	
ω: continuous-time frequency (rad/sec) Frequency spectrum may be infinite in extent and aperiodic.	$\hat{\omega}$: continuous-time frequency (rad/sec) Frequency spectrum is periodic (repeats every 2π)

Overview of remaining topics

Domain	Topic	Discrete Time	Continuous Time
Time	Signals	*SPFirst* Ch. 4 ✓	*SPFirst* Ch. 2 ✓
	Systems	*SPFirst* Ch. 5	*SPFirst* Ch. 9
	Convolution	*SPFirst* Ch. 5	*SPFirst* Ch. 9
Frequency	Fourier series	** ✓	*SPFirst* Ch. 3 ✓
	Fourier transforms	*SPFirst* Ch. 6	*SPFirst* Ch. 11
	Frequency response	*SPFirst* Ch. 6	*SPFirst* Ch. 10
Generalized Frequency	z / Laplace Transforms	*SPFirst* Ch. 7-8	Supplemental Text
	Transfer Functions	*SPFirst* Ch. 7-8	Supplemental Text
	System Stability	*SPFirst* Ch. 8	*SPFirst* Ch. 9
Mixed Signal	Sampling	➔ *SPFirst* Ch. 4	*SPFirst* Ch. 12

[11:35-11:40]

The Nyquist rate ($2\ f_{\max}$) is different from the Nyquist frequency ($f_s/2$)

What happens if $f_s = 2f_{max}$? Example:

$$x(t) = \cos(2\pi f_0 t)\,, \qquad -\infty < t < \infty$$

$$x[n] = x(t)|_{t=nT_s} = \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

Let $f_0 = \frac{1}{2} f_s$. Then,

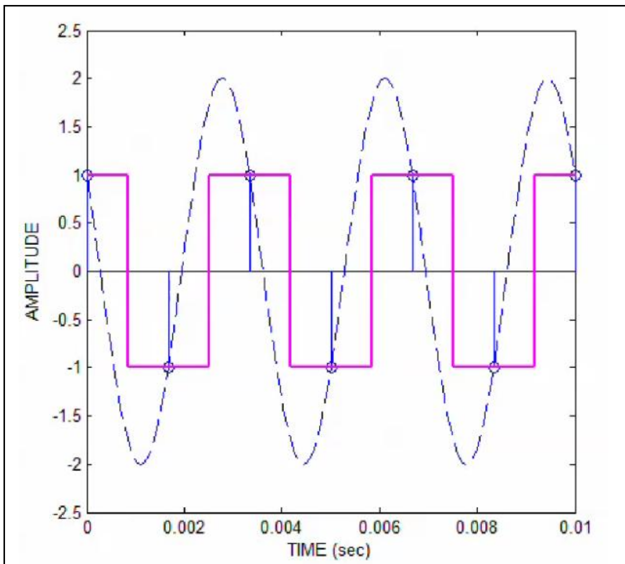
$$x[n] = \cos\left(2\pi \frac{\frac{1}{2} f_s}{f_s} n\right) = \cos(\pi n) = (-1)^n$$

What about $y(t) = \sin(2\pi f_0 t)$ when $f_0 = \frac{1}{2} f_s$?

$$y[n] = \sin(\pi n) = 0$$

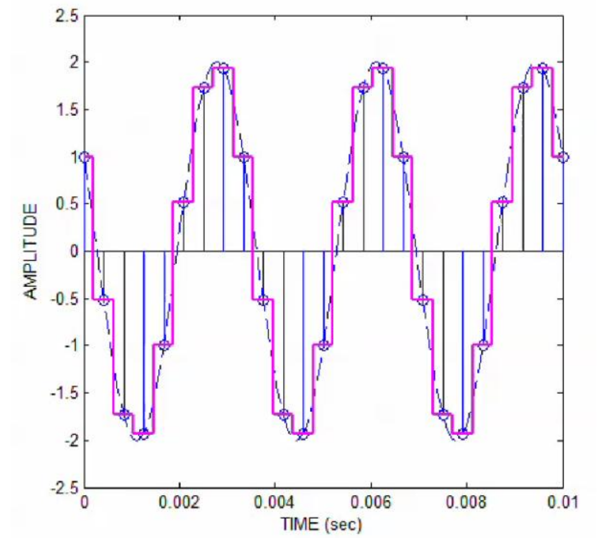
So when $f_s = 2f_0$, a cosine makes it through the sampling process but sine does not.

[11:50-12:10] Demo: sampling and reconstructing a sinusoid



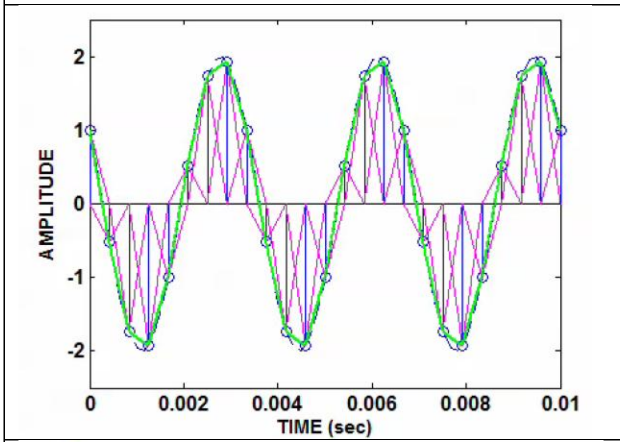
Reconstruction with a square pulse
Sampling near the Nyquist rate

- Captures the correct number of zero crossings
- Amplitude is reduced
- Shape is not captured



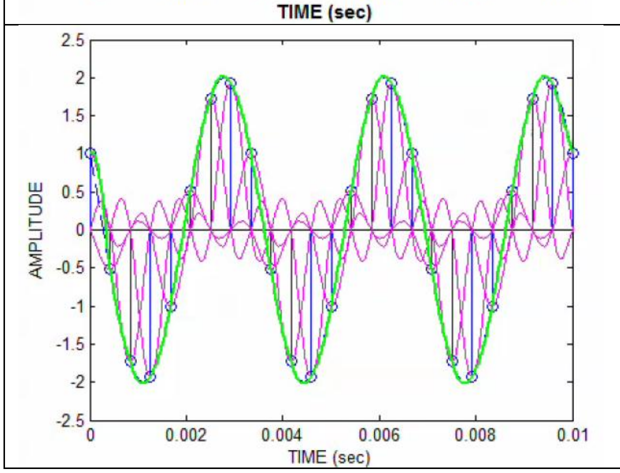
Reconstruction with a square pulse
Sampling at >8x the Nyquist rate

- Accurately tracks the amplitude
- Shape is closer to a sinusoid



Reconstruction with triangular pulse
Sampling at >8x the Nyquist rate

- Shape improves even more



Reconstruction with truncated sinc pulse
Sampling at >8x the Nyquist rate

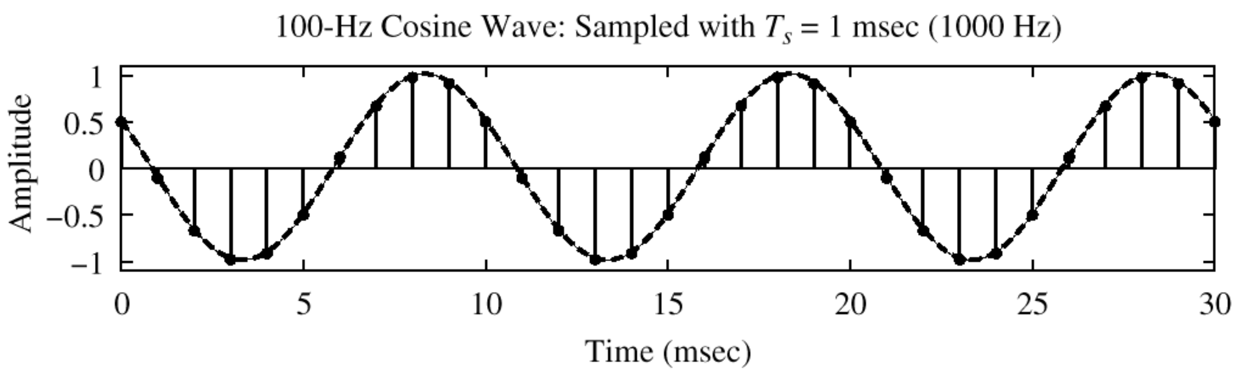
- Shape is nearly perfect
- Amplitude is nearly perfect

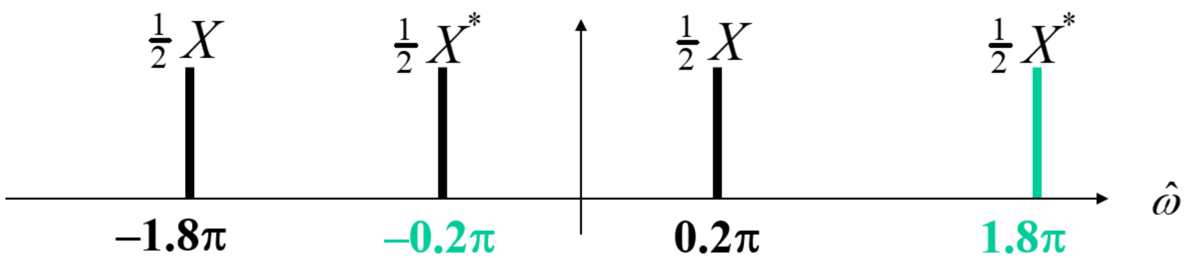
[12:10-] Spectrum of discrete-time signal

A continuous time sinusoid contains frequencies at $\pm f_0$.

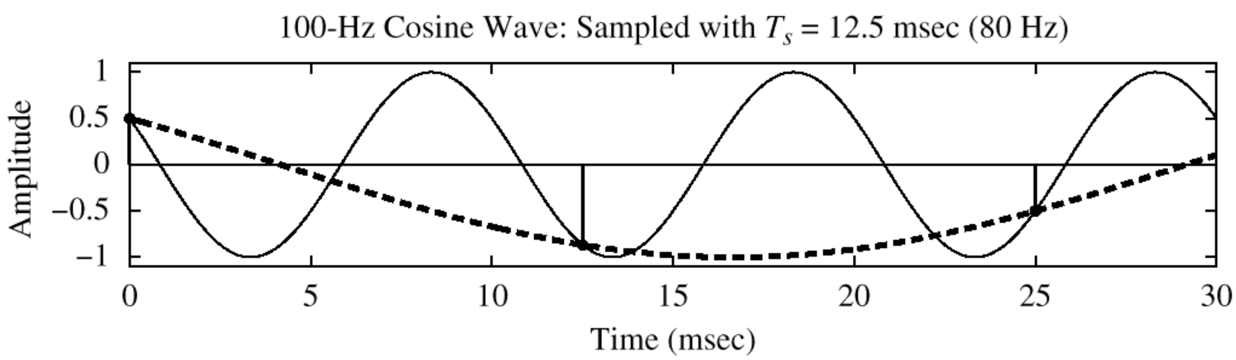
A discrete-time sinusoid also contains all aliases $\hat{\omega} = \frac{\omega_0}{f_s} + 2\pi\ell$ for $\ell = 0, \pm 1, \pm 2, \ldots$

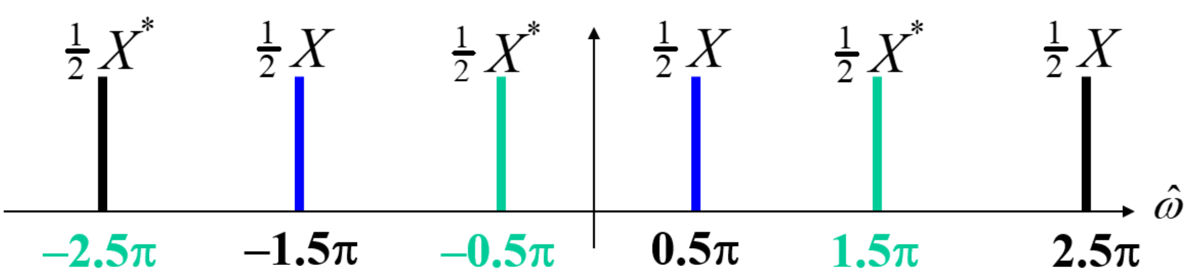
Example #1: $f_0 = 100$ Hz, $f_s = 1000$ Hz (Oversampling)



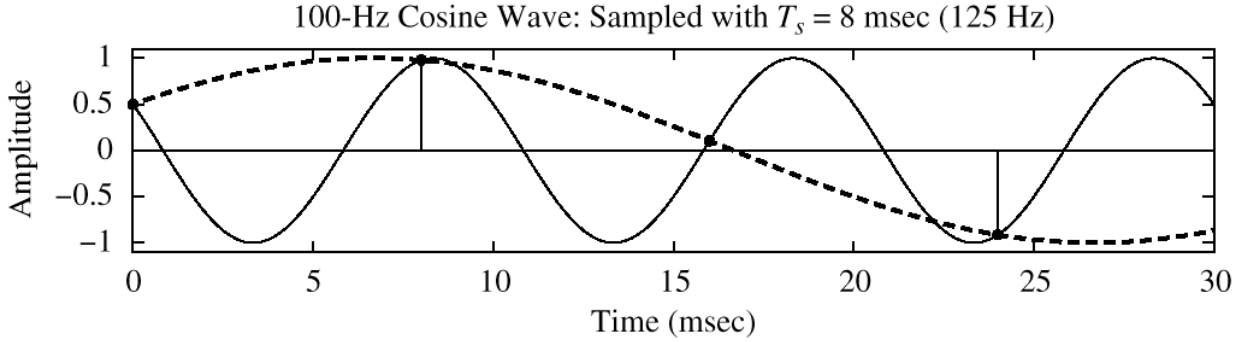


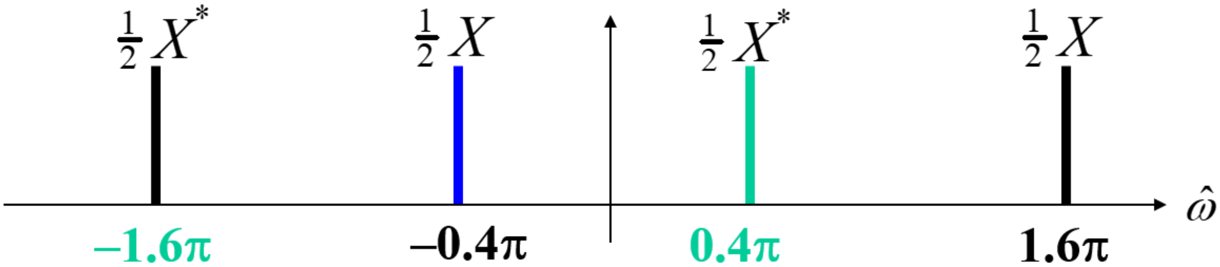
Example #2: $f_0 = 100$ Hz, $f_s = 80$ Hz (Aliasing by undersampling)





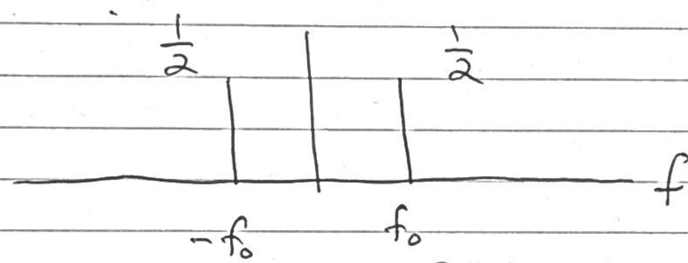
Example #3: $f_0 = 100$ Hz, $f_s = 80$ Hz (Folding by undersampling)





Review – Sampling 9/23/25

$x(t) = \cos(2\pi f_0 t)$ for $-\infty < t < \infty$



$\cos(2\pi f_0 t) = \frac{1}{2} e^{-j 2\pi f_0 t} + \frac{1}{2} e^{j 2\pi f_0 t}$

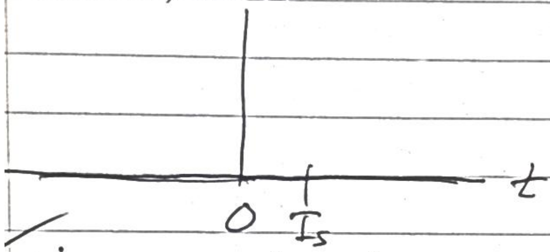
Sampling Theorem – sample at

$f_s > 2 f_{max} \rightarrow f_s > 2 f_0$

If $f_s > 2 f_0$, then we can reconstruct $x(t)$ from the sampled version, but the Sampling Theorem does not tell us how to do that

Continuous-Time

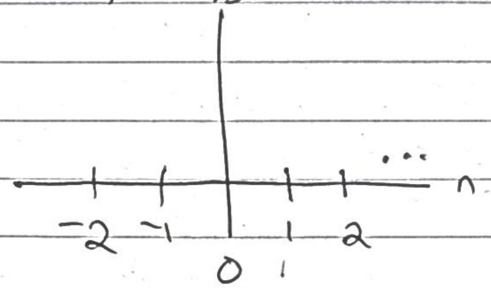
Time Domain



✓ time is real-valued

Discrete-Time

Time Domain

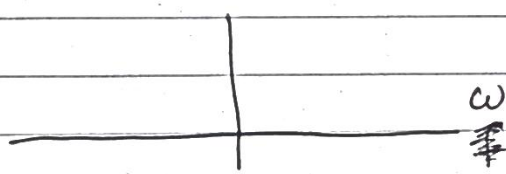


✓ time in integer-valued

Sampling: $t = n T_s$

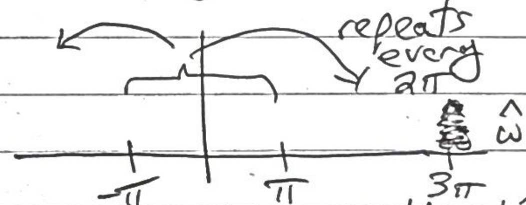
✓ Periodicity will be different — see handout on Discrete-Time Periodicity

Frequency Domain



frequency is real-valued [rad/s]

✓ **Frequency Domain** (Periodicity)



frequency is real-valued [rad/sample]

$$\hat{\omega}_0 = 2\pi \frac{f_0}{f_s}$$

Sampling Theorem: $f_s > 2 f_0 \leftrightarrow f_0 < \frac{1}{2} f_s$

Range of continuous-time freq. captured:

$$-\frac{1}{2} f_s < f < \frac{1}{2} f_s$$

$$-\pi < \hat{\omega} < \pi$$

Slide 6-3

$x(t) = \cos(2\pi f_0 t)$ for $-\infty < t < \infty$

Sample at $f_s \rightarrow t = \frac{n}{f_s}$

$$x[n] = x(t)\Big|_{t=\frac{n}{f_s}} = \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

Let $f_0 = \frac{1}{2} f_s$,

$$x[n] = \cos\left(2\pi \frac{\frac{1}{2} f_s}{f_s} n\right) = \cos(\pi n) = (-1)^n$$

$y(t) = \sin(2\pi f_0 t)$ and let $f_0 = \frac{1}{2} f_s$,

$$y[n] = \sin(\pi n) = 0$$